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Single module identifiability

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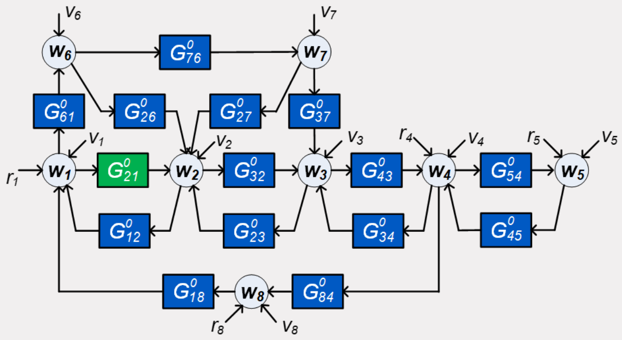
Single module identifiability

Single module identifiability

Situation that we are only interested in **one single module** in the network:

$$w(t) = \underbrace{(I - G)^{-1}R(q)}_{T_{wr}(q)}\, r(t) + \underbrace{(I - G)^{-1}H(q)e(t)}_{\bar{v}(t)}$$



Definition

A module G_{ji} is network identifiable from (w, r) in a model set $\mathcal{M}$ at $M_0 = M(\theta_0)$, if for all $M(\theta_1) \in \mathcal{M}$:

$$\left.\begin{array}{c} T_{wr}(q,\theta_1) = T_{wr}(q,\theta_0) \\ \Phi_{\bar{v}}(\omega,\theta_1) = \Phi_{\bar{v}}(\omega,\theta_0) \end{array}\right\} \Longrightarrow G_{ji}(\theta_1) = G_{ji}(\theta_0)$$

Generic identifiability holds if this is true for *almost all* models $M(\theta_0) \in \mathcal{M}$

Single module identifiability

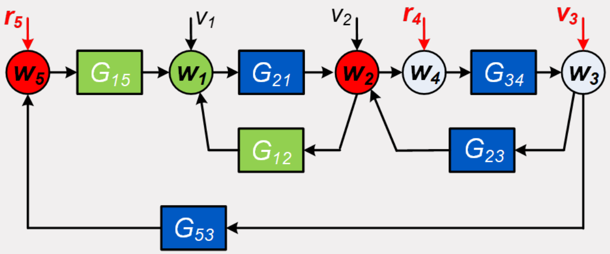
Recall from full network identifiability:

$$\check{T}_j(q,\theta): \qquad [G_{j*}(\theta)\ H_{j*}(\theta)\ R_{j*}(\theta)] = [0\ *\ 0\ 0\ *\ |\ *\ *\ 0\ |\ 1\ 0]$$

$\check{T}_j(q,\theta)$ is the transfer matrix from $\mathcal{X}_j$ to $\mathcal{W}_j^-$, denoted by $T_{\mathcal{W}_j^- \mathcal{X}_j}$

$\mathcal{X}_j$: external signals with non-parametrized entries in $[H_{j\star}(\theta) \quad R_{j\star}(\theta)]$

$\mathcal{W}_j^-$: node signals $\{w_k\}$with parametrized entries in $G_{j\star}(\theta)$



With v_1 and v_2 correlated:

$$\mathcal{X}_1 = \{v_3, r_4, r_5\}$$

$$\mathcal{W}_1^- = \{w_2, w_5\}$$

Single module identifiability

Recall from full network identifiability:

For identifiability of row j: $[G_{j*}(\theta)\ H_{j*}(\theta)\ R_{j*}\theta)]$, the condition is that

$T_{\mathcal{W}_j^- \mathcal{X}_j}$ has full row rank

i.e. every row of $T_{\mathcal{W}_j^- \mathcal{X}_j}$ has an independent contribution

Extending this same reasoning:

Consider $T_{\mathcal{W}_j^- \backslash \{w_i\} \mathcal{X}_j}$ being $T_{\mathcal{W}_j^- \mathcal{X}_j}$ with row i removed

Then the condition for identifiabiltiy of G_{ji} is that

$$\text{rank}\, T_{\mathcal{W}_j^- \mathcal{X}_j} = \text{rank}\, T_{\mathcal{W}_j^- \backslash \{w_i\} \mathcal{X}_j} + 1$$

i.e. row of $T_{\mathcal{W}_j^- \mathcal{X}_j}$ related to w_i has an independent contribution

Single module identifiability

Theorem – single module identifiability

Under conditions on absence of algebraic loops (see full network identifiability case), module G_{ji} is globally identifiable from (w, r) in $\mathcal{M}$ if

$$\text{rank } T_{\mathcal{W}_j^- \mathcal{X}_j}(q,\theta) = \text{rank } T_{\mathcal{W}_j^- \backslash \{w_i\} \mathcal{X}_j}(q,\theta) + 1 \quad \text{for all } M(\theta) \in \mathcal{M}.$$

Interpretation: input w_i should have an excitation component which is independent of the inputs to other parametrized modules that map into w_j

The condition is also **necessary** if

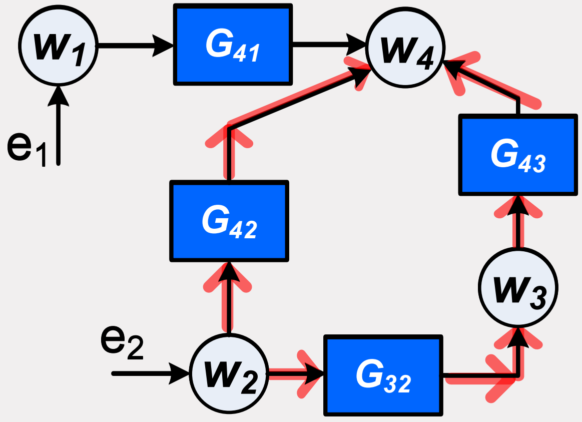
- All parametrized entries in $M(\theta)$ are parametrized independently, and
- Every parametrized entry in $G(\theta)$ is parametrized as an open subset of the set of all LTI systems

Weerts et al, CDC 2018; Hendrickx et al. TAC 2019.

Example

Network with 4 parametrized modules

Which of modules G_{41}, G_{42} can be identified?



$$w = Te, \quad T = (I - G)^{-1}H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & G_{32} \\ G_{41} & G_{42} + G_{32}G_{43} \end{bmatrix}$$

$$\begin{matrix} w_1 \leftarrow \\ w_2 \leftarrow \\ w_3 \leftarrow \end{matrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & G_{32} \end{bmatrix} = \check{T}_4$$

$\uparrow e_1 \quad \uparrow e_2$

Removing first row → Rank reduction → G_{41} identifiable

Removing second row → Rank invariant → G_{42} not-identifiable

Why is G_{42} not-identifiable? There is a parallel path without additional excitation

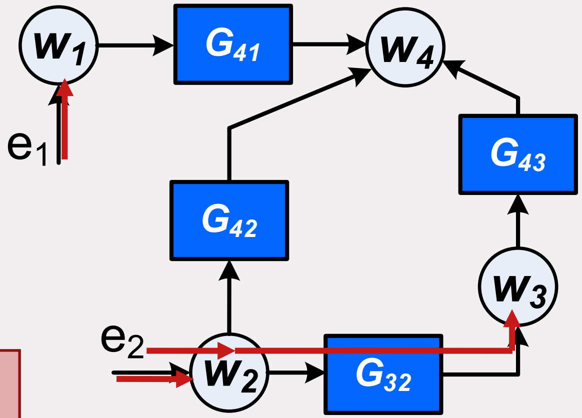
Weerts et al, CDC 2018.

TU/e

Single module identifiability

Checking rank conditons is tedious.

Generically we can check the **vertex disjoint paths**:



$$T_{\mathcal{W}_4\mathcal{X}_4} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & G_{32} \end{bmatrix}$$

$$\underbrace{e_1, e_2}_{\mathcal{X}_4} \rightarrow \underbrace{w_1, w_2, w_3}_{\mathcal{W}_4}$$

2 vertex disjoint paths

$$T_{\mathcal{W}_4\backslash\{w_1\}\mathcal{X}_4} = \begin{bmatrix} 0 & 1 \\ 0 & G_{32} \end{bmatrix}$$

$$\underbrace{e_1, e_2}_{\mathcal{X}_4} \rightarrow \underbrace{w_2, w_3}_{\mathcal{W}_4\backslash\{w_1\}}$$

1 vertex disjoint path

$\Longrightarrow$ G_{41} generically identifiable

Single module identifiability

- **Path-based conditions** (vertex disjoint paths) can be used to verify the generic identifiability conditions

- Well suited for analysis, but less suitable for **synthesis question**:

  "Where to add excitation signals so as to achieve generic identifiabiltiy of a particular module?"

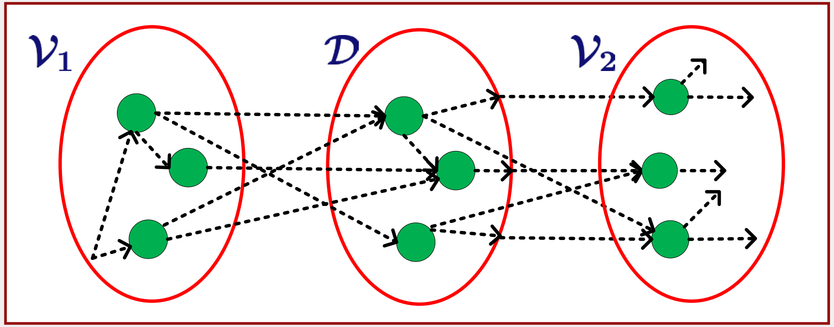
- Therefore: reformulate conditons in terms of disconnecting sets

Single module identifiability

Use of **disconnecting sets**[1],[2]:

A vertex set $\mathcal{D}$ is **disconnecting** vertex sets $\mathcal{V}_1$ and $\mathcal{V}_2$ if upon removal of the vertices in $\mathcal{D}$ there is no directed link from $\mathcal{V}_1$ to $\mathcal{V}_2$.

$\mathcal{D}$ is a **minimum** disconnecting set if it has the smallest cardinality $|\mathcal{D}|$



$\mathcal{D}$ can contain elements of $\mathcal{V}_1$ and/or $\mathcal{V}_2$

Result[1],[2]: max # vertex disjoint paths between $\mathcal{V}_1$ and $\mathcal{V}_2$ =

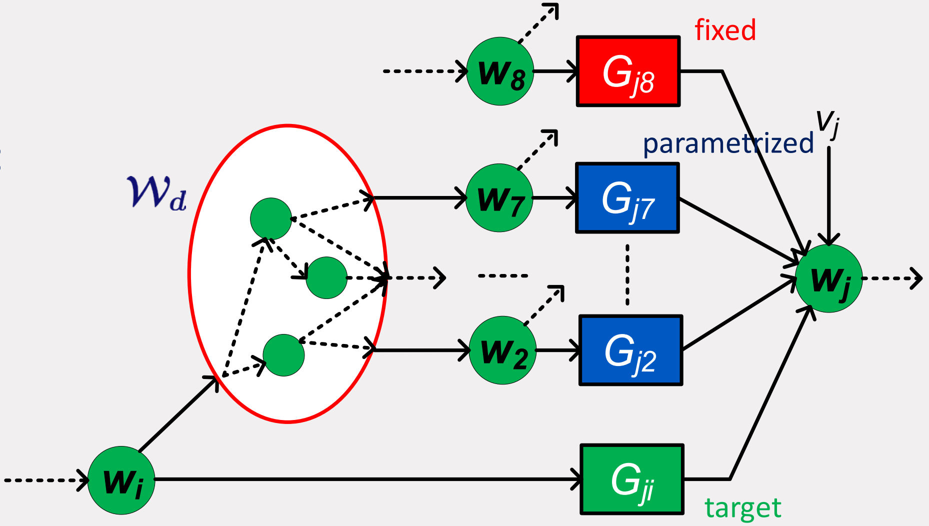
= cardinality of minimum $\mathcal{V}_1 - \mathcal{V}_2$ disconnecting set

[1] Schrijver, 2003.
[2] Hendricks et al., TAC2019

Single module identifiability

Reformulation of the path-based conditions:

Define $\mathcal{W}_d$ as a disconnecting set between w_i and $\mathcal{W}_j^- \setminus \{w_i\}$ such that $w_i \notin \mathcal{W}_d$



Corollary – single module generic identifiability:

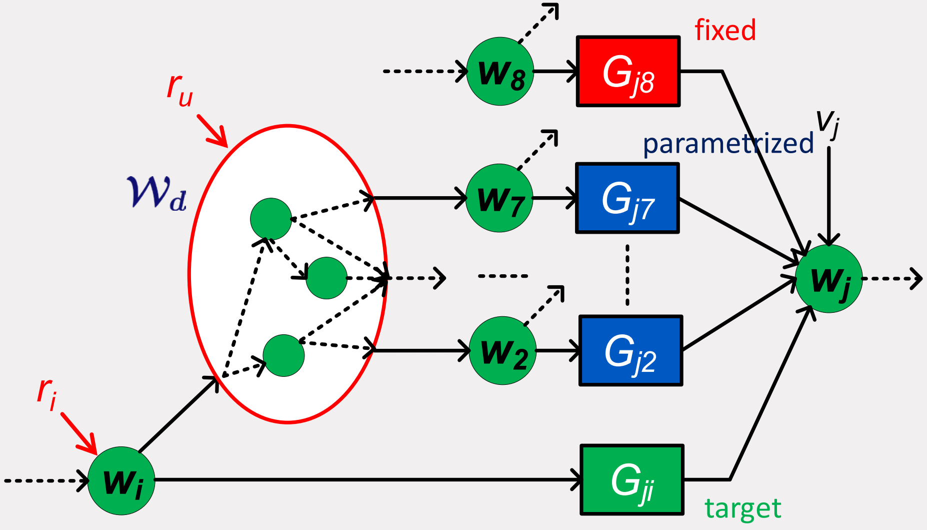
Under similar conditions as presented before, module G_{ji} is generically identifiable from (w, r) in $\mathcal{M}$ if

$$b_{\mathcal{X}_j \to \mathcal{W}_d \cup \{w_i\}} = b_{\mathcal{X}_j \to \mathcal{W}_d} + 1$$

i.e. there are independent excitations to the nodes in $\mathcal{W}_d$ and to w_i

[1] Shi et al., IFAC 2020.

Single module identifiability



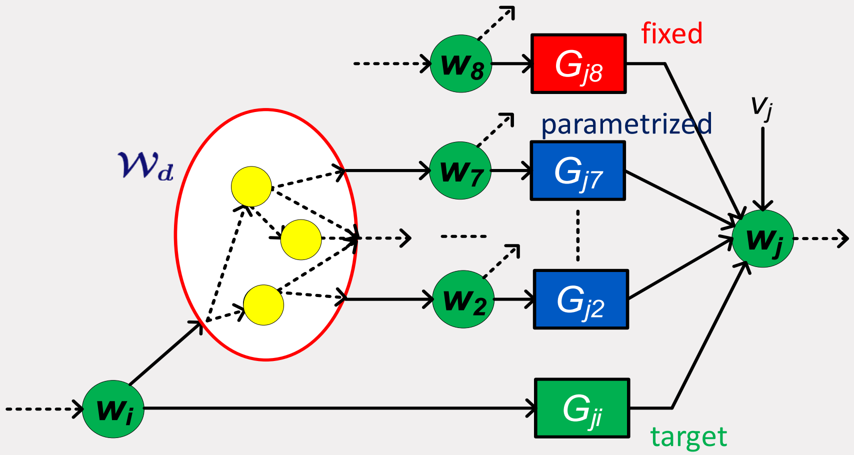
$$b_{\mathcal{X}_j \to \mathcal{W}_d \cup \{w_i\}} = b_{\mathcal{X}_j \to \mathcal{W}_d} + 1$$

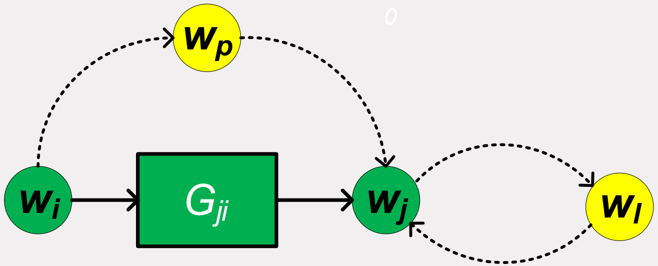
the independent excitations can be added to the nodes in $\mathcal{W}_d$ and w_i directly, or reach them "from a distance" through vertex disjoint paths

This addresses the synthesis question too

[1] Gevers et al., SYSID 2018.
[2] Shi et al., IFAC 2020, Automatica, 2022

Relation between disconnecting set and a PP&L condition

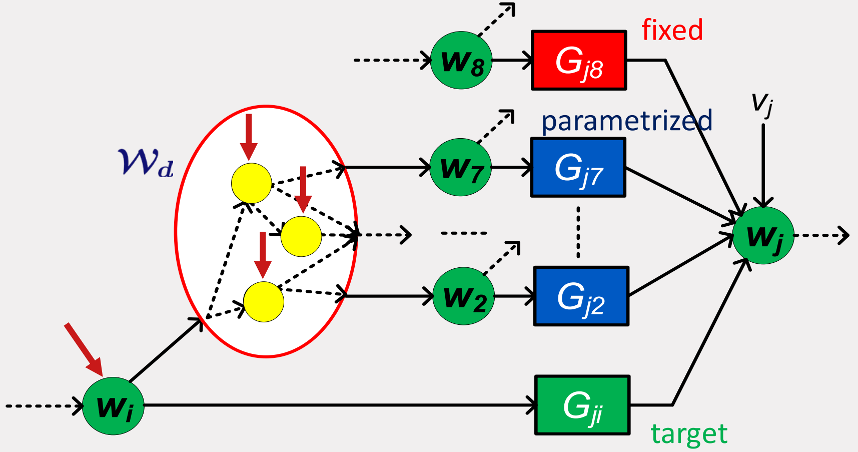


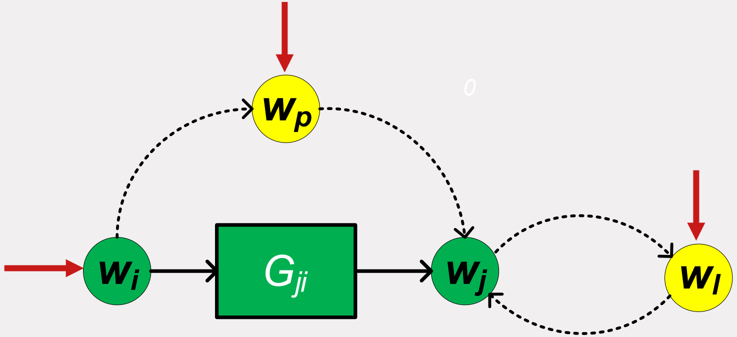


Result:

The nodes in $\mathcal{W}_d$ block all parallel paths from w_i to w_j and all loops around w_j that pass through parametrized modules, different from G_{ji}.

Relation between disconnecting set and a PP&L condition



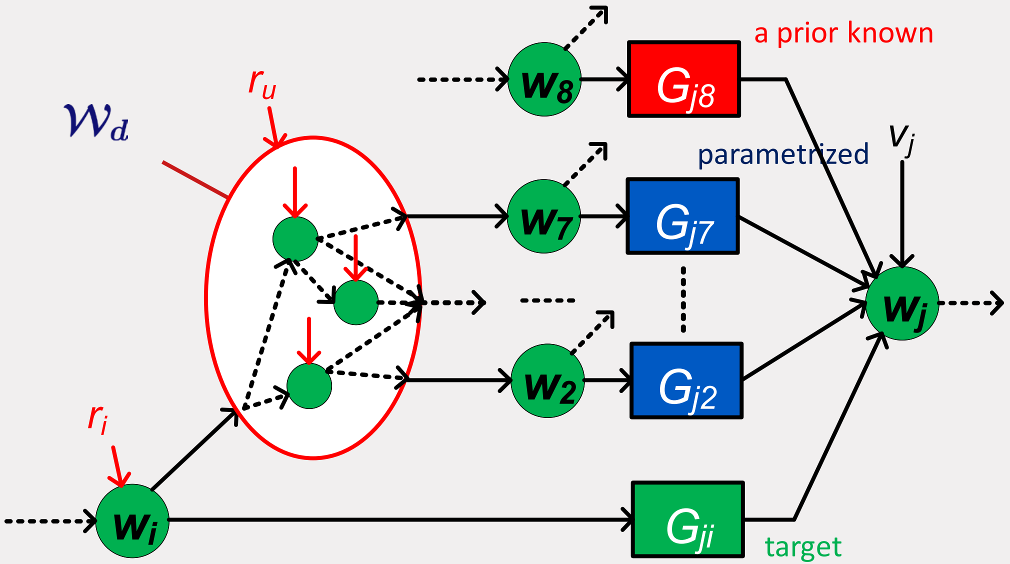


So one option for formulating the excitaton conditions for single module generic identifiability is:
independently excite w_i and a node in each parallel path and loop

TU/e

Single module identifiability

From **analysis** to **synthesis:** Where to allocate excitation signals?



Result[2]**:** G_{ji} is generically identifiable within $\mathcal{M}$ if independent external signals are added to the nodes in $\mathcal{W}_d$ and w_i .

[2] Shi, Cheng, VdHof, IFAC 2020; Automatica 2022.

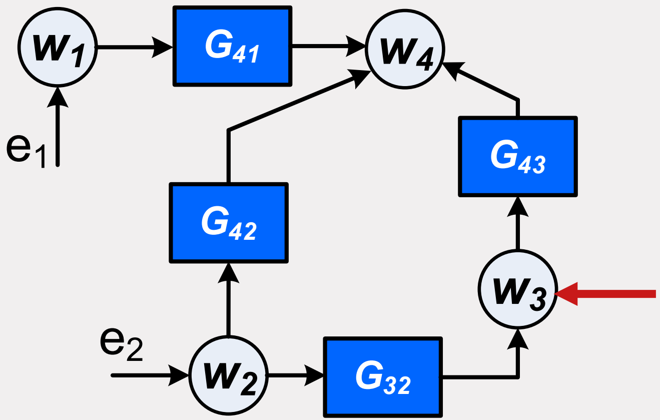
Single module identifiability

Identifiability of G_{42} in model set
with four parametrized modules:

Input of target module is: w_2

Disconnecting set from w_2 to $\{w_3, w_1\}$
is $\{w_3\}$

External signals are required on w_2 and w_3



Single module identifiability – fixed modules

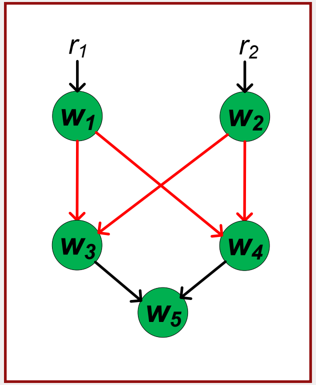
Fixed (non-parametrized) modules can fully be handled in this setting.

However they need to satisfy one additional condition, to avoid that they affect the number of vertex disjoint paths in the parametrized model set

In model set $\mathcal{M}$ the rank of any fixed submatrix of

$$\begin{bmatrix} I - G(\theta) & R(\theta) & H(\theta) \end{bmatrix}$$

(that does not depend on θ), is equal to its structural rank



$$det \begin{bmatrix} G_{31} & G_{32} \\ G_{41} & G_{42} \end{bmatrix} \neq 0$$

structural rank = maximum rank of all matrices with the same nonzero pattern

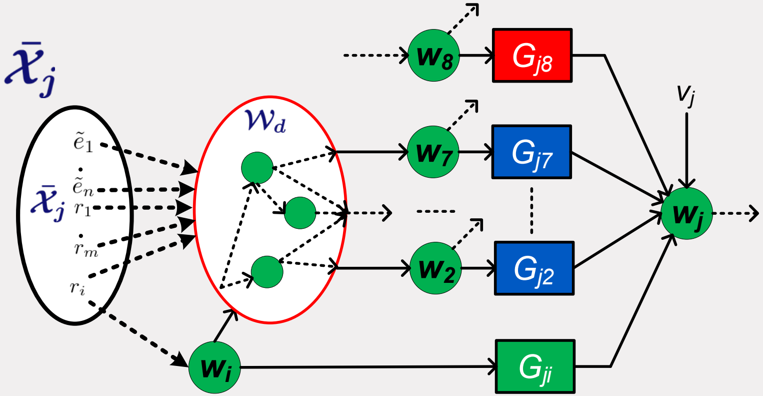
Single module identifiability – K result

Consider a model set $\mathcal{M}$ and a disconnecting set $\mathcal{D}$ from $\bar{\mathcal{X}} \subset \mathcal{X}$ to $\bar{\mathcal{W}} \subset \mathcal{W}$. Then there exists a proper transfer matrix K such that

$$T_{\bar{\mathcal{W}}\bar{\mathcal{X}}} = K T_{\mathcal{D}\bar{\mathcal{X}}}$$

i.e. the transfer matrix $T_{\bar{\mathcal{W}}\bar{\mathcal{X}}}$ can be decomposed through the vertices in $\mathcal{D}$.

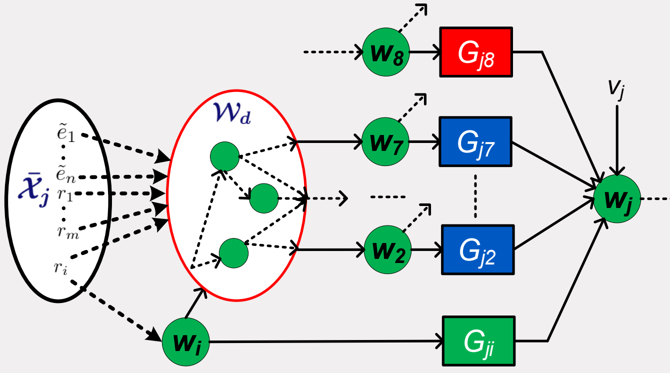
Let $\mathcal{W}_d \cup w_i$ be the disconnecting set satisfying the identifiability conditon for some set of external signals $\bar{\mathcal{X}}_j$



[1] Shi, Cheng, VdH, IFAC 2020; Automatica 2022.

Single module identifiability – indirect method

Let $\mathcal{W}_d \cup \{w_i\}$ be the disconnecting set satisfying the identifiability conditon for some set of external signals $\bar{\mathcal{X}}_j$



Then

$$G_{ji} = T_{j\bar{\mathcal{X}}_j} \begin{bmatrix} T_{i\bar{\mathcal{X}}_j} \\ T_{\mathcal{W}_d\bar{\mathcal{X}}_j} \end{bmatrix}^\dagger [1\, 0\, \cdots\, 0]^T$$

If $\bar{\mathcal{X}}_j$ is composed of measured r-signals only, then the transfer functions on the right hand side can be simply estimated from data $\{w_i, w_j, w_\mathcal{D}, r_{\bar{\mathcal{X}}_j}\}$

Indirect method: estimating transfer functions $\bar{\mathcal{X}}_j \to \mathcal{W}_d \cup \{w_i\}$

$\bar{\mathcal{X}}_j \to w_j$

and taking the quotient to obtain G_{ji}

The disconnecting set shows the flexiblity in selecting the measured signals

Reasoning

From the j-th row of $(I-G)T = X$ and the columns according to $\bar{\mathcal{X}}_j$:

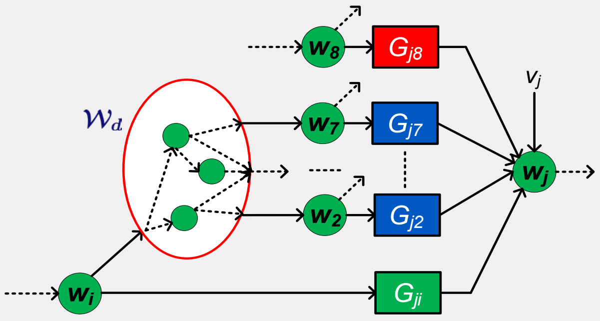
$$\begin{bmatrix} -G_{ji} & -G_{j\mathcal{N}_j^- \backslash w_i} & 1 & 0 \end{bmatrix} \begin{bmatrix} T_{i\bar{\mathcal{X}}_j} \\ T_{\mathcal{N}_j^- \backslash w_i \bar{\mathcal{X}}_j} \\ T_{j\bar{\mathcal{X}}_j} \\ \star \end{bmatrix} = X_{j\bar{\mathcal{X}}_j}$$

$$\begin{bmatrix} -G_{ji} & -G_{j\mathcal{N}_j^- \backslash w_i} K & 1 & 0 \end{bmatrix} \begin{bmatrix} T_{i\bar{\mathcal{X}}_j} \\ T_{\mathcal{W}_d \bar{\mathcal{X}}_j} \\ T_{j\bar{\mathcal{X}}_j} \\ \star \end{bmatrix} = 0$$

$$\begin{bmatrix} G_{ji} & G_{j\mathcal{N}_j^- \backslash w_i} K \end{bmatrix} \begin{bmatrix} T_{i\bar{\mathcal{X}}_j} \\ T_{\mathcal{W}_d \bar{\mathcal{X}}_j} \end{bmatrix} = T_{j\bar{\mathcal{X}}_j}$$

□

Single module identifiability – indirect method



If all required excitation is provided by r-signals only, the target module can be identified on the basis of measured signals $\{w_i, w_j, w_\mathcal{D}, r_{\bar{\mathcal{X}}_j}\}$

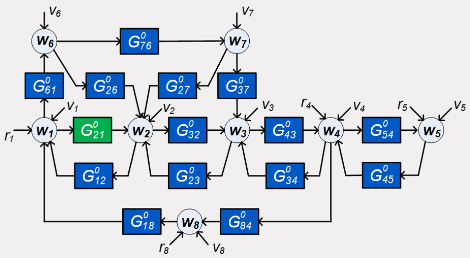
BUT: the analysis provided so far is for the situation of having all w-nodes measured (full measurement case)

Single module identifiability analysis for the situaton of partial measurement requires a separate analysis

Single module identifiability - partial measurement

$$w(t) = \underbrace{(I-G)^{-1}R(q)}_{T_{wr}(q)}\, r(t) + \underbrace{(I-G)^{-1}H(q)e(t)}_{\bar{v}(t)}$$

$w_c(t) = Cw(t)$ with C a (constant) selection matrix



Definition

A module G_{ji} is network identifiable from (w_c, r) in a model set $\mathcal{M}$ at $M_0 = M(\theta_0)$, if for all $M(\theta_1) \in \mathcal{M}$:

$$\left.\begin{array}{r} CT_{wr}(q,\theta_1) = CT_{wr}(q,\theta_0) \\ C\Phi_{\bar{v}}(\omega,\theta_1)C^T = C\Phi_{\bar{v}}(\omega,\theta_0)C^T \end{array}\right\} \Longrightarrow G_{ji}(\theta_1) = G_{ji}(\theta_0)$$

Generic identifiability holds if this is true for *almost all* models $M(\theta_0) \in \mathcal{M}$

[1] Bazanella et al., CDC 2019
[2] Shi et al., ArXiv 2020; TAC 2023.

Single module identifiability - partial measurement

Critical step: Turn information on $C\Phi_{\bar{v}}(\omega)C^T$ into information on $CT_{we}(q)$

Definition:

Two network models $M_i = (G_i, R_i, C_i, H_i, \Lambda_i)$, $i = 1, 2$, are called (observationally) equivalent if

$$C_1 T_1(z) R_1 = C_2 T_2(z) R_2 \quad \text{and} \quad C_1 \Phi_1(z) C_1^T = C_2 \Phi_2(z) C_2^T$$

Theorem (canonical noise model):

For any network model $M = (G, R, C, H, \Lambda)$ and $w = (w_c^T, w_z^T)^T$ there exists an equivalent network model

$$\tilde{M} = (G, R, C, \begin{bmatrix} \tilde{H}^\star & 0 \end{bmatrix}^\star, \tilde{\Lambda})$$

with $\tilde{H}$ square $c \times c$, $c = |\mathcal{C}|$, and $\tilde{H}$ monic, stable and minimum phase

[1] Shi, Cheng, VdH, ArXiv 2020, TAC 2023.

Single module identifiability - partial measurement

The measured node signals $w_{\mathcal{C}}$ can be equivalently described by a network model with
- disturbances at $w_{\mathcal{C}}$ only
- no disturbances at $w_{\mathcal{Z}}$

Theorem (canonical noise model):

For any network model $M = (G, R, C, H, \Lambda)$ and $w = (w_{\mathcal{C}}^T, w_{\mathcal{Z}}^T)^T$ there exists an equivalent network model

$$\tilde{M} = (G, R, C, \begin{bmatrix}\tilde{H}^\star & 0\end{bmatrix}^\star, \tilde{\Lambda})$$

with $\tilde{H}$ square $c \times c$, $c = |\mathcal{C}|$, and $\tilde{H}$ monic, stable and minimum phase

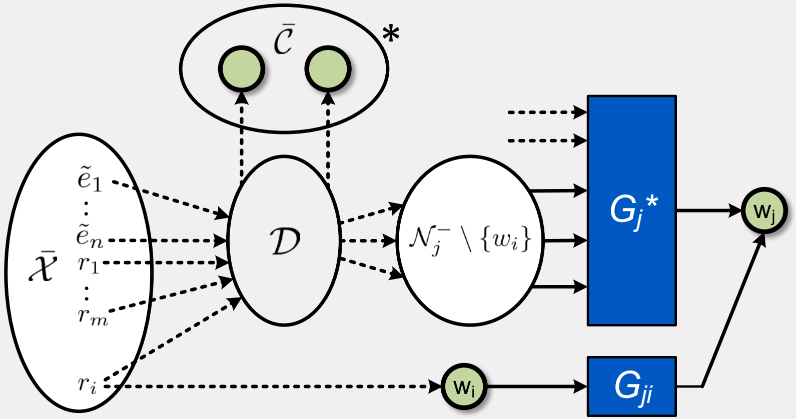
TU/e

Single module identifiability – partial measurements

Theorem

Let $w_i, w_j \in \mathcal{C}$, the set of measured nodes

and $\mathcal{N}_j^-$:
the measured inputs to parametrized modules, and the unmeasured inputs to fixed modules



then G_{ji} is generically identifiable in $\mathcal{M}$ from (w_c, r) if for some $\bar{\mathcal{X}} \subset \mathcal{X}_j$ and $\bar{\mathcal{C}} \subset \mathcal{C} \setminus \{w_i\}$ there exists a disconnecting set $\mathcal{D}$ from $\bar{\mathcal{X}}$ to $\bar{\mathcal{C}} \cup \mathcal{N}_j^- \setminus \{w_i\}$ such that

1. $b_{\bar{\mathcal{X}} \to \mathcal{D} \cup \{w_i\}} = |\mathcal{D}| + 1$
2. $b_{\mathcal{D} \to \bar{\mathcal{C}}} = |\mathcal{D}|$

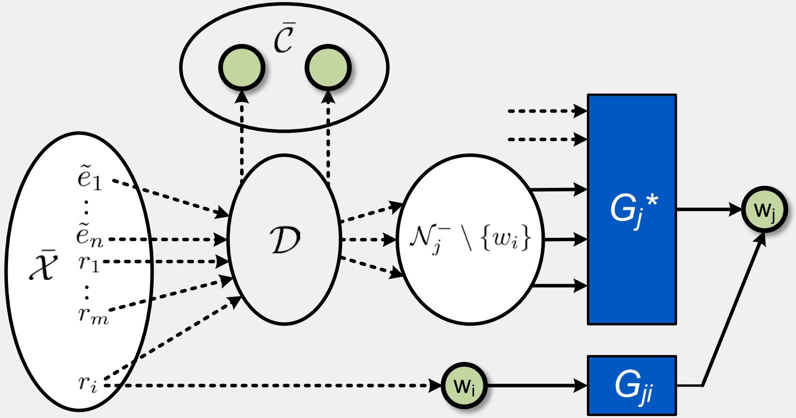
All nodes in $\mathcal{D} \cup \{w_i\}$ need to be independently excited and observed (direct or indirect)

[1] Shi et al., TAC 2023.

* Figure is sketched for the situation $w_j \notin \bar{\mathcal{C}}$

Single module identifiability – partial measurements

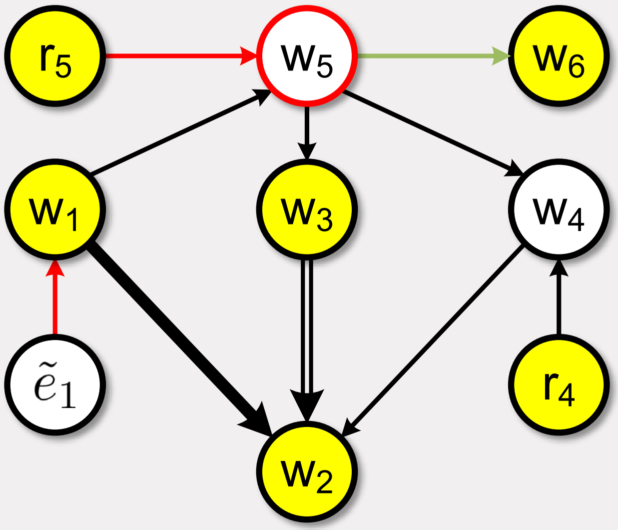
- Results are independent of any identification method
- And can be used for allocating excitation signals
- Extensions are available for situations where w_i and/or w_j are not measured



Single module identifiability – partial measurements

Example

- Target module G_{21}; G_{23} is non-parametrized
- Yellow nodes are measured
- $\mathcal{N}_2^- = \{w_1, w_4\}$
- $\mathcal{X}_2^- = \{\tilde{e}_1, r_4, r_5\}$
- w_5 disconnects r_5 from $\{w_1, w_4\}$
- $\bar{\mathcal{X}} = \{\tilde{e}_1, r_5\}$
- w_5 is observed by $\bar{\mathcal{C}} = \{w_6\}$



Conclusion: G_{21} is generically identifiable from the measured nodes.

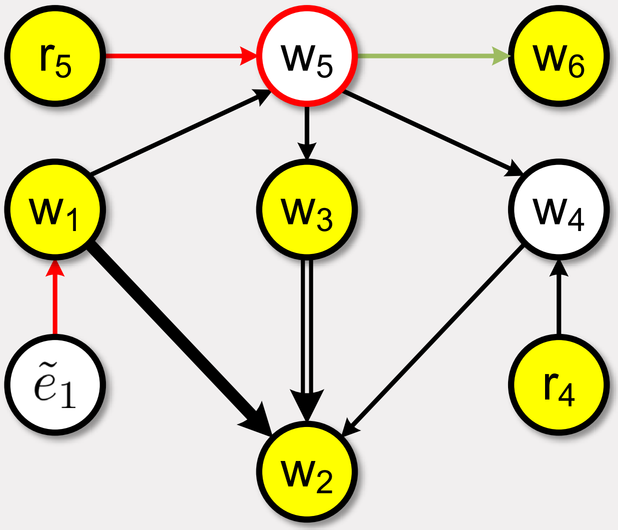
TU/e

Single module identifiability – partial measurements

Example

G_{21} is generically identifiable but:
how to identify it?

- Indirect method:
  requires a measured excitation of w_1
- Direct method:
  requires parallel path and loop condition
- Both methods fail:
  $\Longrightarrow$ generalized method that combines node signals and r signals as predictor inputs[1]



[1] Ramaswamy, VdH, Dankers, CDC 2019